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Optimal Power-Limited Rendezvous with Upper and Lower Bounds on Thrust

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Optimal power-limited rendezvous with variable exhaust velocity is investigated for propulsion systems having both upper and lower bounds on thrust magnitude. In this model the spacecraft thrusters have four admissible states, thrusting at the upper saturation level, unsaturated power-limited thrusting, thrusting at the lower saturation level, and unpowered (i.e., engine off.) A fifth chattering state is also possible. The mathematical structure of the solution of the optimal rendezvous problem associated with this propulsion model is found. Computer simulations of rendezvous with a satellite in circular orbit are presented. For rendezvous near circular orbit four classes of chattering solutions of two types can occur. It is assumed that multiple thrusters can be mounted on the spacecraft and that they can operate independently. Applying the efficiency condition presented in a previous paper, a logic for switching individual thrusters on and off for optimal fuel usage and computational efficiency is presented.

I. Introduction

THE study of trajectories of spacecraft having low-thrust, variable exhaust velocity, power-limited, propulsion systems has existed for some time. After the early work,[1–12] the basic ideas were summarized and compared with those based on constant exhaust-velocity chemically propelled systems by Edelbaum[13] and Marec.[14]

In the past few years there has been renewed interest in optimal trajectories and rendezvous based on variable exhaust-velocity, power-limited propulsion systems.[15–21] Recently Carter[22,23] and Pardis and Carter[24] have included the effects of thrust saturation in the study of power-limited spacecraft rendezvous, noting the similarity to a problem first studied by Letov.[25–28] A more realistic model, suggested by Kechichian,[29] considers both upper and lower bounds on thrust magnitude in power-limited rendezvous problems. This model, in which for planar problems the thrust is restricted to an annular region, has apparently not been studied in detail in the literature on control theory.

The present paper extends the work of Pardis and Carter[24] in applying the efficiency condition and in determining an optimal number of multiple thrusters to efficiently complete a rendezvous mission by avoiding thrust saturation, and applies these ideas to rendezvous problems in which the thruster propulsion models are similar to those of Kechichian.[29] Although the emphasis is on linear systems, many of the ideas presented are also applicable to nonlinear systems. The material is new in the following ways.

1) In the analysis that follows, it is assumed that except for chattering modes a power-limited spacecraft thruster has four admissible states: thrusting at an upper saturation level, unsaturated thrusting, thrusting at a lower saturation level, and unpowered (i.e., engine off). Kechichian's[29] model assumes the first three of these possible states. The propulsion model assumed herein then consists of hard upper and lower constraints on thrust magnitude and the capability of instantly switching to or from an unpowered state. The latter capability has a profound effect on the nature of solutions to certain boundary-value problems that result from the necessary conditions for optimal thrusting. One such effect is the introduction of chattering solutions for certain boundary conditions.

2) A new result that follows from the aforementioned assumptions and the necessary conditions for an optimal solution is that a thruster will exhibit lower saturation if the primer vector magnitude is between the lower saturation bound and one-half of this bound; if the primer vector magnitude falls below one-half of the lower saturation bound, then the thruster should cut off and the vehicle should coast until the primer magnitude rises above this level. If the primer vector magnitude should remain exactly at one-half of the lower saturation bound, chattering solutions can appear.

3) Applying the necessary conditions derived herein to the problem of rendezvous of a spacecraft with a satellite in circular orbit using the Clohessy–Wiltshire equations, the results of a computer simulation of a trajectory that exhibits the first three of the four aforementioned thrusting states is presented. A simulation that depicts all of the four thrusting states is then shown. It is found also that exactly four classes of chattering solutions of two distinct types are possible for this application.

4) The efficiency condition, first presented formally in Ref. 24, is readily extended to the problem discussed here.

5) Finally, it is assumed that a spacecraft can be powered by identical multiple thrusters that can be independently cut on or off during flight. The efficiency conditions is then applied in this situation. A theorem is then presented that determines whether or not a multiple-thruster unsaturated solution exists for specified initial and terminal conditions. If such a solution exists, switching logics can easily be determined that allow the mission to be performed without thruster saturation by switching certain thrusters on and off in sequence throughout the flight.

This results in the maximum fuel efficiency for the mission, and, for linear systems, a problem that can be solved, at least symbolically, in closed form. A switching logic and flight-path simulation is presented graphically for the case of a spacecraft near a satellite in circular orbit.

II. Structure of the Solution

Some of the ideas in Ref. 24 are applied to a power-limited rendezvous problem in which the thrust is constrained in a manner similar to that suggested by Kechichian.[29]

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A. Mathematical Model

The equations of motion of a spacecraft relative to a satellite are of the general form

$$x'(\theta) = g[\theta, \boldsymbol{x}(\theta)] + B(\theta)u(\theta) \tag{1}$$

Here θ represents time or true anomaly or any convenient independent variable contained in a closed, bounded interval $\Theta = [\theta_0, \theta_f]$ of real numbers. An element $\theta \in \Theta$ will be referred to as an instant. The prime denotes differentiation with respect to θ; $\boldsymbol{x}(\theta)$ is a state vector in an open region X of $\mathfrak{R}^n$; g is a function defined on the Cartesian product $\Theta \times X$ that is continuous in θ and continuously differentiable in x; B is a continuous function mapping Θ into the set of real $n \times m$ matrices, where m and n are positive integers; and u is an element of a set $\mathcal{U}(U)$ of admissible controls (i.e., a set of Lebesgue measurable functions that map Θ into a specified subset U of $\mathfrak{R}^m$). The points x_0 and x_f in X define the fixed end conditions

$$\boldsymbol{x}(\theta_0) = x_0, \qquad \boldsymbol{x}(\theta_f) = x_f \tag{2}$$

Typically $u(\theta)$ represents the applied thrust of a spacecraft, $\boldsymbol{x}(\theta)$ consists of three position and three velocity coordinates, so that $n = 6$ and $m = 3$; if the motion is restricted to an orbital plane there are two position and two velocity coordinates, so that $n = 4$ and $m = 2$.

It has been shown by early researchers that for variable exhaust-velocity, power-limited propulsion systems, the index of performance to be minimized is an integral of the square of the applied thrust magnitude over the flight interval. For some problems, such as the case where θ represents true anomaly and Eq. (1) is given by the Tschauner–Hempel equations, a weighting factor may be needed in the integrand.[20] The index of performance is, therefore, given by the Lebesgue integral

$$J[u] = \int_{\theta_0}^{\theta_f} \gamma(\theta)\boldsymbol{u}(\theta)^T\boldsymbol{u}(\theta)\,\mathrm{d}\theta \tag{3}$$

where γ is a positive real-valued continuous function on Θ. The superscript T is used to represent the transpose of a vector or matrix, and the symbol $|\cdot|$ represents the magnitude of a vector, specifically, $|\boldsymbol{u}(\theta)| = [\boldsymbol{u}(\theta)^T\boldsymbol{u}(\theta)]^{1/2}$.

Letting U denote a closed subset of $\mathfrak{R}^m$, the set $\mathcal{U}(U)$ of admissible controls is defined as the set of all Lebesgue measurable functions mapping Θ into U. Any statement that is true on a Lebesgue measurable set except on a subset of Lebesgue measure zero will be said to be true almost everywhere (a.e.) on that set. Historically, researchers in power-limited rendezvous considered U to be $\mathfrak{R}^3$ or, for planar problems, $\mathfrak{R}^2$. Recently Carter[22,23] and Pardis and Carter[24] defined U as a closed ball in $\mathfrak{R}^m$, and Kechichian[29] set U to be a closed region bounded by concentric spheres in $\mathfrak{R}^3$ or a closed annular region in $\mathfrak{R}^2$. In the present paper U is a closed annular region with the inclusion of the point at the origin for $m = 2$ or its obvious generalization to a closed region bounded by concentric spheres in m space. The annular region or its obvious generalization represents thrust with upper and lower bounds on its magnitude; the point at the origin represents an unpowered state. Lebesgue measurability of an admissible control admits the idealization of instantly shutting a thruster on or off with arbitrary rapidity and is a natural setting for controls that may involve chatter.

The idealized optimal power-limited rendezvous problem can now be stated as follows. Find an idealized thrust function u from the set $\mathcal{U}(U)$ of admissible controls that minimizes the index of performance $J[u]$ as defined by Eq. (3) subject to the differential equation (1) and the fixed end conditions (2). Unfortunately, solutions to this problem do not exist for all reachable end conditions (2) because the set U is not convex. However, in 1962 Warga presented an existence theorem[30] and necessary conditions for an optimal solution[31] of a more general class of problems obtained by embedding the class of admissible controls into a more general class of probability measure-valued functions called relaxed controls. In this work,[30,31] he showed that an optimal relaxed solution can be approximated arbitrarily closely by an original admissible control. This type of approximating admissible control typically oscillates rapidly among certain points of U if no admissible solution exists, and has come to be known as chatter. Viewed geometrically, the consequence of embedding the set of original admissible controls in the set of relaxed controls effectively replaces the set U by its closed convex hull and, for this problem, the function u^Tu that appears in the integrand of Eq. (3) is replaced by its convex envelope so that in the integrand u^Tu is replaced by the cone $b_1|u|$ over the set $|u| \le b_1$ in the convex hull of U. A solution of the relaxed problem that can be geometrically represented by control values in the convex hull of U but not in U itself on a set of positive Lebesgue measure is called a strictly relaxed regime. It is known that strictly relaxed regimes are always singular.[32] Strictly relaxed regimes are called sliding regimes or chattering regimes. Surveys of the literature and illustrative examples on chattering and singular controls with emphasis on aerospace problems have been done by Marchal[33] and Marchal and Contensou.[34] For the problem considered here optimal strictly relaxed, or chattering controls for the planar case are effectively represented as optimal singular controls taking values on the convex hull of the annulus U strictly between the origin and the circle determine by the lower bound on the thrust magnitude.

B. Form of an Optimal Thrusting Function

The minimum principle of Pontryagin establishes necessary conditions for an optimal thrust function. Because of the constraint U an admissible control is restricted by $\boldsymbol{u}(\theta) = 0$ or $b_1 \le |\boldsymbol{u}(\theta)| \le b_2$, $(\theta_0 \le \theta \le \theta_f)$. By abuse of notation, the Hamiltonian associated with this problem may be written as

$$H(u) = \lambda_0\gamma(\theta)u^Tu + \lambda(\theta)^T\{g[\theta, \boldsymbol{x}(\theta)] + B(\theta)u\} \tag{4}$$

where $\lambda_0 \ge 0$ and $\lambda\colon \Theta \to \mathfrak{R}^n$ is the adjoint or costate vector function. The part of the Hamiltonian that depends on u is

$$L(u) = \lambda_0\gamma(\theta)u^Tu + \lambda(\theta)^TB(\theta)u \tag{5}$$

It is necessary that $\boldsymbol{u}(\theta)$ be selected in U to minimize $L(u)$ a.e. on Θ.

A problem is called abnormal if $\lambda_0 = 0$; otherwise it is called normal and λ_0 is positive. Although the investigation of abnormal problems is not more difficult than that of normal problems, it is somewhat special, and will not be pursued in the present paper. For normal problems λ_0 can be set equal to any positive number without loss of generality. We, therefore, set $\lambda_0 = \frac{1}{2}$ for convenience.

With this adjustment, the part of the Hamiltonian (4) that depends on u can be written as

$$L(u) = \gamma(\theta)\left[\frac{1}{2}u^Tu + \frac{\lambda(\theta)^T}{\gamma(\theta)}B(\theta)u\right] \tag{6}$$

Appearing in this expression is the ubiquitious primer vector

$$\boldsymbol{p}(\theta) = \frac{B(\theta)^T}{\gamma(\theta)}\lambda(\theta) \tag{7}$$

that has a profound effect on the nature of optimal solutions. It is necessary that a.e. on Θ, $\boldsymbol{u}(\theta)$ be selected in U to minimize

$$L(u) = \gamma(\theta)\left[\tfrac{1}{2}u^Tu + p(\theta)^Tu\right] \tag{8}$$

If $\boldsymbol{u}$ is not the zero vector, we may write

$$\boldsymbol{u} = \boldsymbol{e}_u f \tag{9}$$

where f is the magnitude of $\boldsymbol{u}$ and $\boldsymbol{e}_u$ is a unit vector in the direction of $\boldsymbol{u}$. If $\boldsymbol{u}$ is the zero vector then Eq. (9) is valid where $\boldsymbol{e}_u$ is an arbitrary vector. Inserting Eq. (9) in Eq. (8), it is necessary that $\boldsymbol{e}_u = -\boldsymbol{p}(\theta)/|\boldsymbol{p}(\theta)|$ whenever $\boldsymbol{p}(\theta) \ne 0$, and $f = 0$ whenever $\boldsymbol{p}(\theta) = 0$. The expression (8), therefore, becomes

$$L\left(\frac{-\boldsymbol{p}(\theta)f}{|\boldsymbol{p}(\theta)|}\right) = \gamma(\theta)\left[\frac{1}{2}f^2 - |\boldsymbol{p}(\theta)|f\right] \tag{10}$$

It is necessary to minimize this expression a.e. on Θ over the set

$$f = 0 \qquad \text{or} \qquad b_1 \le f \le b_2$$

This minimization is accomplished by

$$f = \begin{cases} b_2, & |p(\theta)| > b_2 \\ |p(\theta)|, & b_1 \le |p(\theta)| \le b_2 \\ b_1, & b_1/2 \le |p(\theta)| < b_1 \\ 0, & |p(\theta)| \le b_1/2 \end{cases} \tag{11}$$

This is seen by completing the square in Eq. (10) and observing the four situations for the vertex of the parabola in relation to the bounds b_1 and b_2.

It can be observed from Eq. (11) that the expression (10) does not have a unique minimum if $|p(\theta)| = b_1/2$. If this occurs on a set of positive Lebesgue measure, then an optimal solution is a relaxed singular solution and may involve chattering on this set. The possibility of equality in the last part of Eq. (11) can be dropped in solutions that do not involve relaxed singular controls. We do this and consider separately the possibility of relaxed singular regimes including chatter.

Utilizing Eq. (9), the expression (11) shows that it is necessary a.e. on Θ for an optimal thruster to consist of four states

$$u(\theta) = \begin{cases} \dfrac{-p(\theta)b_2}{|p(\theta)|}, & |p(\theta)| > b_2 \\ -p(\theta), & b_1 \le |p(\theta)| \le b_2 \\ \dfrac{-p(\theta)b_1}{|p(\theta)|}, & \dfrac{b_1}{2} \le |p(\theta)| < b_1 \\ 0, & |-p(\theta)| < \dfrac{b_1}{2} \end{cases} \tag{12}$$

and a possible additional relaxed singular regime that may contain chatter, where $|p(\theta)| = b_1/2$. It is convenient to represent this information in the form

$$u(\theta) = -h(c, \theta)p(\theta) \tag{13}$$

a.e. on Θ, where $c = \lambda(\theta_0)$, and the four states are incorporated into the function

$$h(c, \theta) = \begin{cases} \dfrac{b_2}{|p(\theta)|}, & |p(\theta)| > b_2 \\ 1, & b_1 \le |p(\theta)| \le b_2 \\ \dfrac{b_1}{|p(\theta)|}, & \dfrac{b_1}{2} \le |p(\theta)| \le b_1 \\ 0, & |-p(\theta)| < \dfrac{b_1}{2} \end{cases} \tag{14}$$

or if $|p(\theta)| = b_1/2$ on a set of positive Lebesgue measure, then a fifth relaxed singular state will occur on this set that may include chatter, and $h(c, \theta)$ is undefined on this set.

A thorough understanding of chatter involves relaxed controls.[30,31] Relaxing the controls effectively replaces the set U by its convex hull and the function $u^T u$ appearing in Eqs. (4–6) and (8) by the function $b_1|u|$ on the set $0 < |u| < b_1$. The expression $u^T u$ is unchanged on the original set U. Strictly relaxed or chattering controls are effectively control functions whose images are in the convex hull of U but not in U itself (i.e., in the set $0 < |u| < b_1$). For this problem, a control chatters on a measurable subset of Θ if and only if

$$|p(\theta)| = b_1/2 \qquad \text{and} \qquad 0 < |u(\theta)| < b_1$$

a.e. on this set, where $u(\theta)$ here denotes the effective control having values in the convex hull of U. If $|p(\theta)| = b_1/2$ and $u(\theta) = 0$ or $|u(\theta)| = b_1$ on a set of positive measure, then this control is not strictly relaxed on the set. It can be viewed, however as a singular relaxed control (i.e., a singular control that is not strictly singular for the problem obtained by replacing U by its convex hull).[32] If the set where $|p(\theta)| = b_1$ is of Lebesgue measure zero, then this set has no effect on the final solution of the problem; for this reason the possibility of equality was removed in Eqs. (12) and (14) so that u is a function mapping Θ into U a.e. on Θ, except on solutions having relaxed singular regimes.

The crucially important primer vector is determined through Eq. (7) by c and the differential equations

$$\lambda'(\theta) = -g_x[\theta, x(\theta)]^T \lambda(\theta) \tag{15}$$

now standard in control theory, where the subscript x denotes differentiation with respect to the vector x, the second argument of g. If u is an optimal thrusting function without relaxed singular regimes, it is necessary that Eqs. (1), (7), and (13–15) are satisfied a.e. on Θ subject to the fixed end conditions (2).

C. Linear Equations of Motion

In many situations the spacecraft is near enough to the rendezvous point that a linearized analysis is useful. In this case $g[\theta, x(\theta)] = A(\theta)x(\theta)$ in Eq. (1) and the adjoint system (15) becomes

$$\lambda'(\theta) = -A(\theta)^T \lambda(\theta) \tag{16}$$

We shall assume that the linearized version of Eq. (1) or, equivalently, the pair (A, B) is controllable on Θ. This part of the work is very similar to Ref. 24.

For the cost function (3) and linear differential equations, Pontryagin's principle provides both necessary and sufficient conditions for an optimal relaxed solution because relaxing the controls effectively replaces the set U by its convex hull.[30–32]

Letting $\Psi(\theta)$ denote any fundamental matrix solution associated with Eq. (16), any solution of (16) can be written as $\lambda(\theta) = \Psi(\theta)c$. With the introduction of the matrix

$$R(\theta) = \Psi(\theta)^T B(\theta) \tag{17}$$

the primer vector (7) may be written as

$$p(\theta) = \frac{R(\theta)^T c}{\gamma(\theta)} \tag{18}$$

and any solution of a linearized version of Eq. (1) satisfying the initial condition in Eq. (2) can be written as

$$x(\theta) = \Psi(\theta)^{-T}\left[\Psi(\theta_0)x_0 + \int_{\theta_0}^{\theta} R(\tau)u(\tau)\,\mathrm{d}\tau\right] \tag{19}$$

where the superscript $-T$ indicates the transpose of the inverse of a matrix. It is convenient to transform $x(\theta)$ to the pseudostate vector

$$z(\theta) = \Psi(\theta)^T x(\theta) - \Psi(\theta_0)^T x_0 \tag{20}$$

so that the boundary conditions are represented by the vector

$$z_f = \Psi(\theta_f)^T x_f - \Psi(\theta_0)^T x_0 \tag{21}$$

and Eq. (19) is transformed to

$$z(\theta) = \int_{\theta_0}^{\theta} R(\tau)u(\tau)\,\mathrm{d}\tau \tag{22}$$

that is subject to the boundary condition

$$z(\theta_f) = z_f \tag{23}$$

Substituting an optimal thrusting function (13) into Eq. (22) one obtains

$$z(\theta) = -N(c, \theta_0, \theta)c \tag{24}$$

where

$$N(c, \theta_0, \theta) = \int_{\theta_0}^{\theta} \frac{h(c,\tau)}{\gamma(\tau)} R(\tau)R(\tau)^T\,\mathrm{d}\tau \tag{25}$$

The vector c is crucial in defining the primer vector (18); the optimal thrusting function (13), which holds except when solutions involve relaxed singular regimes; and the optimal pseudostate trajectory (22). For boundary values not having relaxed singular regimes in the solutions, the two-point boundary-value problem can be succinctly stated as the problem of finding a solution vector c of the nonlinear equation

$$-N(c, \theta_0, \theta_f)c = z_f \tag{26}$$

Relaxed singular regimes appear on a set of positive measure if and only if the primer vector (18) has a magnitude of $b_1/2$ a.e. on that set. Chattering occurs if and only if, additionally, the control is strictly relaxed. A necessary and sufficient condition for chattering on a set is, therefore, that $\boldsymbol{c}^T R(\theta)R(\theta)^T \boldsymbol{c} = \gamma(\theta)^2 b_1^2/4$, and $0 < |\boldsymbol{u}(\theta)| < b_1$ a.e. on the set, where $\boldsymbol{u}(\theta)$ represents an effective control value (i.e., a control value on the convex hull of U). If $A(\theta)$, $B(\theta)$, and $\gamma(\theta)$ are analytic on Θ, then a solution having chatter on any infinite subset of Θ is a relaxed singular solution everywhere on Θ, and in this case an optimal solution cannot consist of both chattering and nonchattering regimes.

III. Efficiency Condition and Multiple Thrusters

The viewpoint taken in this paper is that it is cost effective to design spacecraft and thrusters so that several thrusters can be mounted on a spacecraft. Rather than design thrusters to satisfy mission requirements, if possible the mission requirements should determine an optimal number of identical off-the-shelf thrusters to be mounted on a spacecraft.

This approach was recently suggested by Pardis and Carter[24] in conjunction with two simple mission design criteria called the existence condition and the efficiency condition, which may be useful in determining, respectively, whether a target point may be reached and whether it may be reached in a fuel-efficient way without thrust saturation.

For the spacecraft thruster model considered in the present paper, the existence condition is identical to that of the previous paper,[24] with the stipulation that the thrust bound b of that paper be the outer thrust saturation level b_2 in the present paper. The efficiency condition, however, makes use of both the inner bound b_1 and the outer bound b_2 as is shown herein.

The efficiency condition follows from the simple idea that if we compare solutions of the minimization problem defined herein for sets U_1 and U_2 in $\mathfrak{R}^m$, where $U_1 \subseteq U_2$, then

$$\min_{u \in \mathcal{U}(U_2)} J[u] \leq \min_{u \in \mathcal{U}(U_1)} J[u]$$

In simple terminology, this says that a set of admissible controls with more constraints can never yield a solution more efficient than one with fewer constraints. Going a step farther, it says that a solution with constraints is never more efficient than a solution without constraints.

A. Application of the Efficiency Condition

The efficiency condition requires that both upper and lower saturation be avoided. This obviously includes the avoidance of solutions involving chatter as well. Except for the trivial situation where $\boldsymbol{u}(\theta)$ is identically zero, this translates to the condition

$$b_1 \leq |\boldsymbol{p}(\theta)| \leq b_2, \qquad \theta \in \Theta \tag{27}$$

because the primer vector $\boldsymbol{p}$ is continuous. It follows from Eq. (14), therefore, that the efficiency condition requires that $h(\boldsymbol{c}, \theta) = 1$, $\theta \in \Theta$, and Eq. (13) requires that $\boldsymbol{u}(\theta) = -\boldsymbol{p}(\theta)$ a.e. on Θ.

For the linear version of the problem, an optimal solution that is totally unsaturated from above or below means that Eqs. (24), (25), and (26) can be replaced, respectively, by

$$z(\theta) = -M(\theta_0, \theta)\boldsymbol{c} \tag{28}$$

$$M(\theta_0, \theta) = \int_{\theta_0}^{\theta} \frac{R(\tau)R(\tau)^T}{\gamma(\theta)} \, \mathrm{d}\tau \tag{29}$$

$$-M(\theta_0, \theta_f)\boldsymbol{c} = \boldsymbol{z}_f \tag{30}$$

It is known that controllability of the pair (A, B) implies that $M(\theta_0, \theta)$ is invertible for $\theta > \theta_0$, so, in view of Eq. (18), the efficiency condition becomes

$$b_1 \leq \frac{\left|R(\theta)^T M(\theta_0, \theta_f)^{-1} \boldsymbol{z}_f\right|}{\gamma(\theta)} \leq b_2, \qquad \theta \in \Theta \tag{31}$$

It is best expressed as follows:

$$b_1 \leq \min_{\theta_0 \leq \theta \leq \theta_f} \frac{\left|R(\theta)^T M(\theta_0, \theta_f)^{-1} \boldsymbol{z}_f\right|}{\gamma(\theta)} \leq \max_{\theta_0 \leq \theta \leq \theta_f} \frac{\left|R(\theta)^T M(\theta_0, \theta_f)^{-1} \boldsymbol{z}_f\right|}{\gamma(\theta)} \leq b_2 \tag{32}$$

Given a nonzero boundary value $\boldsymbol{z}_f$, an optimal solution will be most fuel efficient if the parameters b_1 and b_2 are far enough apart so that Eq. (32) holds. Additionally, the computational complexity is minimized in this situation, because a closed-form solution is found in this case. One can view the inequality (32) as a description of the set of boundary values $\boldsymbol{z}_f$ whose solution is devoid of both saturation and chatter.

B. Multiple-Thruster Unsaturated Solutions

For a specific mission, the efficiency condition requires that the lower thrust saturation bound b_1 and the upper thrust saturation bound b_2 be far enough apart that inequality (32) is satisfied. According to the viewpoint of this paper, the thrusters are designed before the mission requirements are specified, so b_1 and b_2 are constant, and for the vector $\boldsymbol{z}_f$ defined by the mission requirements, inequality (32) may not be satisfied. For this reason, it may be advantageous to use multiple thrusters.

1. Multiple-Thruster Efficiency Condition

An extensive investigation of the multiple-thruster power-limited rendezvous problem is presented elsewhere[35] and will not be discussed here. The aspect that is presented here is an application of the efficiency condition in the presence of multiple thrusters. For this reason, the only multiple-thruster solutions considered in this paper are unsaturated solutions. Chattering solutions involving multiple thrusters, for example, are a digression here but are examined elsewhere.[35]

From Eqs. (22) and (23), it follows that the linearized minimization problem requires the minimization of $J[u]$ as defined by Eq. (3) over the set of admissible controls $\mathcal{U}(U)$ subject to the constraint

$$\int_{\theta_0}^{\theta_f} R(\theta)\boldsymbol{u}(\theta) \, \mathrm{d}\theta = \boldsymbol{z}_f \tag{33}$$

Now consider the use of multiple thrusters, and let the positive integer ν denote the number of identical thrusters mounted on the spacecraft. Under the assumption that all of the thrusters are linked so that they must burn simultaneously, the new optimization problem is identical to the original problem with the exception that $\boldsymbol{u}(\theta)$ is replaced by $\nu\boldsymbol{u}(\theta)$ in Eq. (33). Therefore, one seeks the minimization of $J[u]$ over $\mathcal{U}(U)$ subject to the constraint

$$\int_{\theta_0}^{\theta_f} R(\theta)\boldsymbol{u}(\theta) \, \mathrm{d}\theta = \frac{\boldsymbol{z}_f}{\nu} \tag{34}$$

Replacing $\boldsymbol{z}_f$ by $\boldsymbol{z}_f/\nu$ in Eq. (32), the new efficiency condition becomes

$$\nu b_1 \leq \min_{\theta_0 \leq \theta \leq \theta_f} \frac{\left|R(\theta)^T M(\theta_0, \theta_f)^{-1} \boldsymbol{z}_f\right|}{\gamma(\theta)} \leq \max_{\theta_0 \leq \theta \leq \theta_f} \frac{\left|R(\theta)^T M(\theta_0, \theta_f)^{-1} \boldsymbol{z}_f\right|}{\gamma(\theta)} \leq \nu b_2 \tag{35}$$

This shows that the upper bound can be effectively increased by increasing ν (i.e., by implementing more thrusters). Unfortunately this implementation also raises the effective lower bound, possibly reducing fuel efficiency as a result of lower saturation.

What is obviously needed for fuel efficiency is more thrusters when the primer vector is near its maximum magnitude, and fewer thrusters when the primer vector is near its minimum magnitude. This calls for a capability of switching thrusters on and off independently during flight to satisfy the efficiency condition, if possible. Assuming that the capability exists to approximate the instantaneous shutting on or off of thrusters independently, conditions under which the efficiency condition can be satisfied are investigated here.

2. Investigation of Multiple-Thruster Switching

It will be assumed that ν is the number of identical thrusters that are mounted on the spacecraft and that these thrusters can be shut on or off independently of one another. It will also be assumed that all thrusters that are burning at any specific instant contribute equally to the total thrust. Since we are investigating intervals where the thrust is unsaturated, condition (12) establishes that $\boldsymbol{u}(\theta) = -\boldsymbol{p}(\theta)$ a.e. on these intervals. For simplicity we shall drop the notation of Lebesgue measure in this part of the study and assume that $\boldsymbol{u}(\theta) = -\boldsymbol{p}(\theta)$ everywhere on an interval without saturation.

Consider an integer-valued function $j: \Theta \to \{0, 1, 2, \ldots, \nu\}$ that describes the number of functioning thrusters at each instant. At any instant $\theta \in \Theta$, let $\boldsymbol{u}(\theta)$ denote the thrust contribution of each of the $j(\theta)$ functioning thrusters. On any interval where $j(\theta)$ is constant, these thrusters satisfy the constraint

$$b_1 \le |\boldsymbol{u}(\theta)| \le b_2$$

The total thrust available at any instant is, therefore, $j(\theta)\boldsymbol{u}(\theta)$, where

$$j(\theta)b_1 \le j(\theta)|\boldsymbol{u}(\theta)| \le j(\theta)b_2 \tag{36}$$

There is some choice in the selection of the function j. The primary criterion is that the efficiency condition be satisfied, if possible. Secondary criteria can be set by the mission planner or engineer.

The function j satisfies the efficiency condition if and only if

$$j(\theta)|\boldsymbol{u}(\theta)| = \frac{\left|R(\theta)^T M(\theta_0, \theta_f)^{-1} \boldsymbol{z}_f\right|}{\gamma(\theta)}, \qquad \theta \in \Theta \tag{37}$$

Substituting this into Eq. (36), it is seen that a multiple-thruster unsaturated solution exists if and only if j can be selected such that

$$j(\theta)b_1 \le \frac{\left|R(\theta)^T M(\theta_0, \theta_f)^{-1} \boldsymbol{z}_f\right|}{\gamma(\theta)} \le j(\theta)b_2, \qquad \theta \in \Theta \tag{38}$$

This expression is seen to be equivalent to

$$\frac{\left|-R(\theta)^T M(\theta_0, \theta_f)^{-1} \boldsymbol{z}_f\right|}{\gamma(\theta)b_2} \le j(\theta) \le \frac{\left|-R(\theta)^T M(\theta_0, \theta_f)^{-1} \boldsymbol{z}_f\right|}{\gamma(\theta)b_1}, \qquad \theta \in \Theta \tag{39}$$

This establishes a proof of the following theorem. To shorten the notation in this theorem and its consequences, the primer vector for an unsaturated solution is written as $\boldsymbol{p}(\theta; \boldsymbol{z}_f)$, where

$$\boldsymbol{p}(\theta; \boldsymbol{z}_f) = \frac{R(\theta)^T M(\theta_0, \theta_f)^{-1} \boldsymbol{z}_f}{\gamma(\theta)} \tag{40}$$

Theorem: A multiple-thruster unsaturated solution exists if and only if for each $\theta \in \Theta$ there is an integer on the closed interval $[|\boldsymbol{p}(\theta; \boldsymbol{z}_f)|/b_2, |\boldsymbol{p}(\theta; \boldsymbol{z}_f)|/b_1]$.

Two conditions that are relatively easy to check follow from this theorem.

Corollary 1: A multiple-thruster unsaturated solution exists if

$$\frac{1}{b_1} - \frac{1}{b_2} \ge \frac{1}{\min_{\theta_0 \le \theta \le \theta_f} |\boldsymbol{p}(\theta; \boldsymbol{z}_f)|} \tag{41}$$

Corollary 2: A multiple-thruster unsaturated solution does not exist if

$$0 < \min_{\theta_0 \le \theta \le \theta_f} |\boldsymbol{p}(\theta; \boldsymbol{z}_f)| < b_1 \tag{42}$$

Corollaries 1 and 2 demonstrate two extremes. If

$$\frac{1}{b_1} - \frac{1}{b_2} < \frac{1}{\min_{\theta_0 \le \theta \le \theta_f} |\boldsymbol{p}(\theta; \boldsymbol{z}_f)|} \le \frac{1}{b_1}$$

then a multiple-thruster unsaturated solution may or may not exist. The theorem will determine which situation occurs.

IV. Spacecraft Rendezvous Near Circular Orbit

The preceding material is applied to a specific example, that of rendezvous of a spacecraft with a satellite in circular orbit.

A. Linearized Equations and Solutions

The following development is an abbreviated version of previous work.[24] The equations of motion of a spacecraft near a satellite in a circular orbit can be represented by the well-known Hill–Clohessy–Wiltshire equations and put in state variable form

$$\begin{aligned}
x_1'(\theta) &= x_4(\theta) \\
x_2'(\theta) &= x_5(\theta) \\
x_3'(\theta) &= x_6(\theta) \\
x_4'(\theta) &= 2x_5(\theta) + u_1(\theta) \\
x_5'(\theta) &= 3x_2(\theta) - 2x_4(\theta) + u_2(\theta) \\
x_6'(\theta) &= -x_3(\theta) + u_3(\theta)
\end{aligned} \tag{43}$$

where the state variables have been normalized by dividing by the constant $k = R_c^3/(\mu m_0)$. This constant k depends on the radius R_c of the orbit of the satellite, the universal gravitational constant times the mass of the planet μ, and the mass m_0 of the spacecraft. This normalization makes all of the variables nondimensional, but any desired units can be obtained by multiplying the appropriate variables by k, where k is specified in the desired units.

It is apparent from Eq. (43) that $n = 6$ and $m = 3$, and $A(\theta)$ and $B(\theta)$ are constant matrices. The solution of Eq. (16) is straightforward and known, and its fundamental matrix solution $\Psi(\theta)$ is presented in previous work.[24] From $\Psi(\theta)$ the matrix $R(\theta)$ is obtained from Eq. (17). For rendezvous near circular orbit there is no need for the weighting function in the cost function (3), and so $\gamma(\theta)$ is set equal to 1. The primer vector (18) simplifies and, except for relaxed singular solutions, the vector $\boldsymbol{c}$ is found numerically through an iterative solution of the nonlinear equation (26). If the magnitude of the primer vector is not identically $b_1/2$, then the optimal thrusting function is determined from Eqs. (13) and (14).

B. Chattering Solutions

In this investigation, the controls are relaxed[30,31] so that the set U is replaced by its convex hull and the function $u^T u$ appearing in the integrand (3) is replaced by $b_1|u|$ on the set $0 < |u| < b_1$. In this context a relaxed solution chatters if and only if it is strictly singular. Since $\gamma(\theta)$ is identically one, the primer vector (18) becomes

$$\boldsymbol{p}(\theta) = R(\theta)^T c \tag{44}$$

where $R(\theta)^T$ consists of the lower three rows of $\Psi(\theta)$ in this case, and $\Psi(\theta)$ is analytic for the problem defined by Eq. (43). A solution is singular on a set of positive measure if and only if

$$\boldsymbol{p}(\theta)^T \boldsymbol{p}(\theta) = b_1^2/4 \tag{45}$$

a.e. on that set; it is strictly singular and chatters if and only if additionally

$$0 < |u(\theta)| < b_1$$

a.e. on that set. Since $R(\theta)$ is analytic on Θ, Eq. (45) has at most finitely many roots on Θ or else it is satisfied identically on Θ. The latter holds if and only if the entire solution is singular. For this reason, any solution having a chattering regime on a set of positive measure is singular identically on Θ.

The condition of identically satisfying Eq. (45) on Θ is analogous to a similar condition in previous papers on singular solutions to problems of fuel-optimal rendezvous near circular orbit,[36,37] and a similar analysis to that found in those papers applies here. Earlier work on singular solutions was also discussed by Marec.[14]

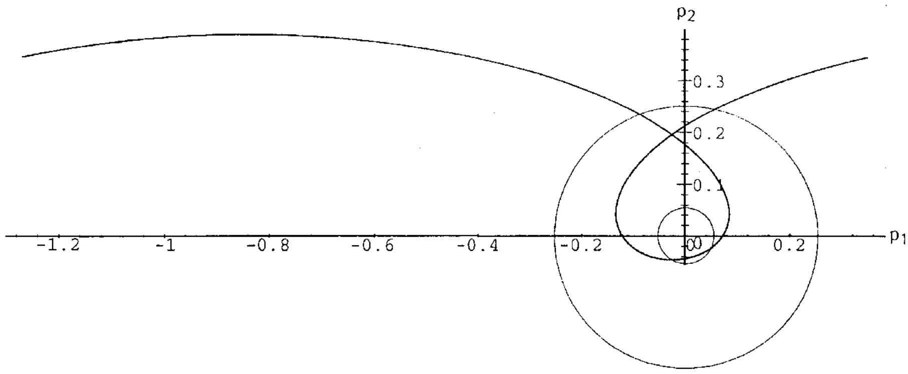

Fig. 1a Primer vector and circular bounds on thrust: $x_1(0) = 0, x_2(0) = 0.2, b_1 = 0.055$, and $b_2 = 0.250$.

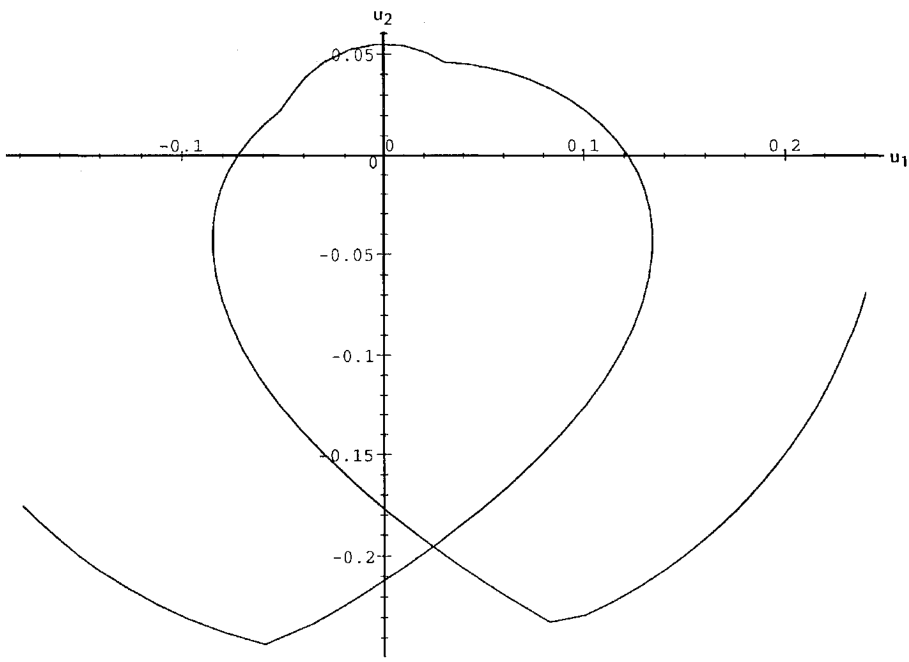

Fig. 1b Optimal thrust with upper and lower saturation: $x_1(0) = 0, x_2(0) = 0.2, b_1 = 0.055$, and $b_2 = 0.250$.

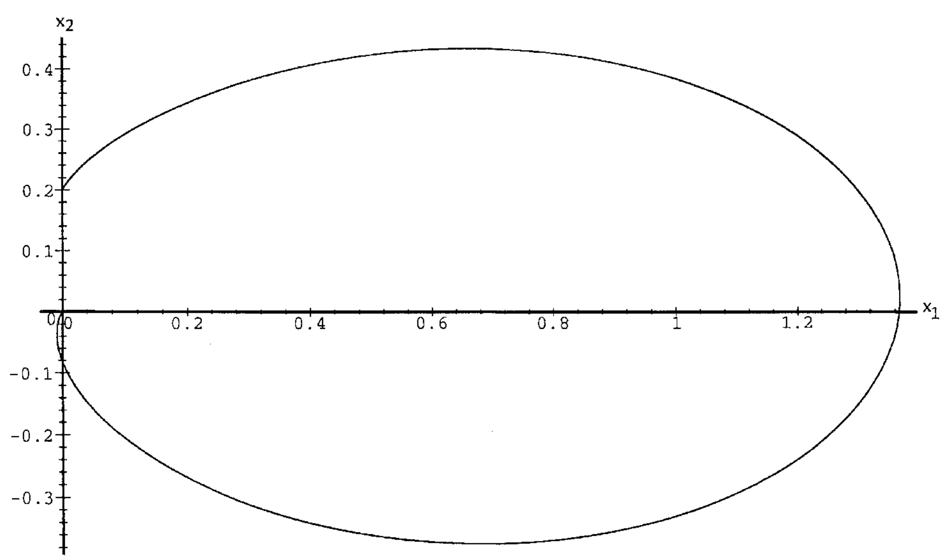

Fig. 1c Flight path: $x_1(0) = 0, x_2(0) = 0.2, b_1 = 0.055$, and $b_2 = 0.250$.

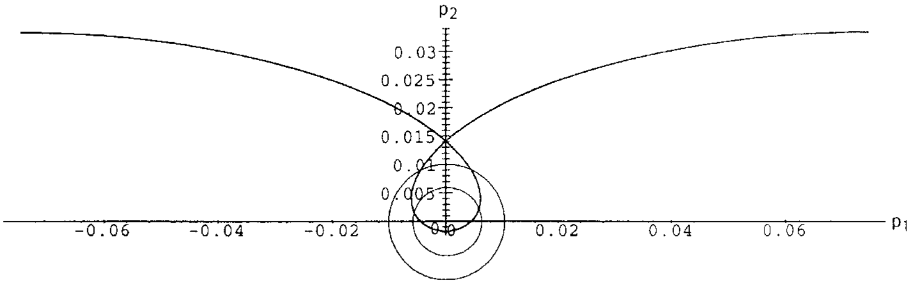

Fig. 2a Primer vector and circular bounds on thrust: $x_1(0) = 0.2, x_2(0) = 0, b_1 = 0.006$, and $b_2 = 0.010$.

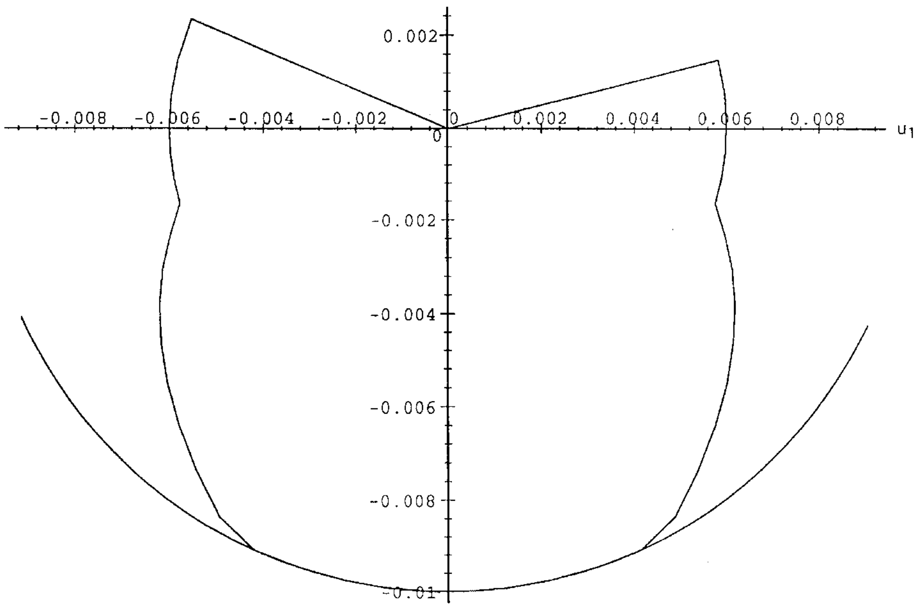

Fig. 2b Optimal thrust with upper saturation, lower saturation, and coast: $x_1(0) = 0.2, x_2(0) = 0, b_1 = 0.006$, and $b_2 = 0.010$.

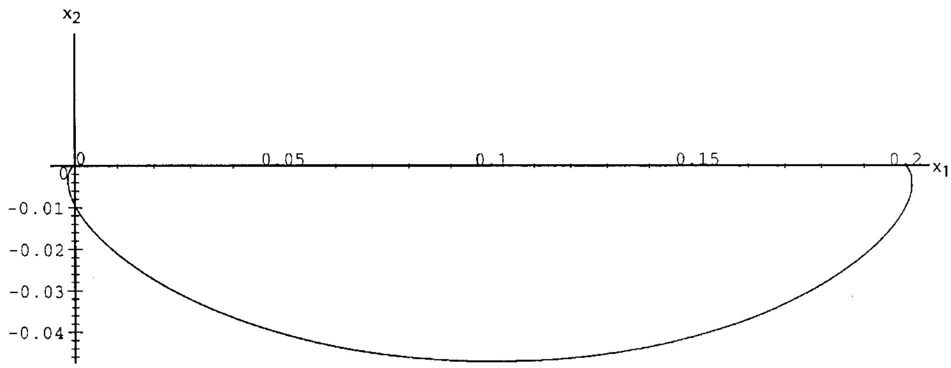

Fig. 2c Flight path: $x_1(0) = 0.2, x_2(0) = 0, b_1 = 0.006$, and $b_2 = 0.010$.

Using $R(\theta)^T$ as determined from previous work,[24] the components of the primer vector (44) are

$$\begin{aligned} p_1(\theta) &= 3c_1\theta + c_2 - 2c_3\sin\theta + 2c_4\cos\theta \\ p_2(\theta) &= 2c_1 - c_3\cos\theta - c_4\sin\theta \\ p_3(\theta) &= c_5\cos\theta - c_6\sin\theta \end{aligned} \tag{46}$$

Combining and replacing the constants c_3, c_4, c_5, and c_6 by new constants $\rho \geq 0$, ψ, α, and β, the system (46) can be put in the form

$$\begin{aligned} p_1(\theta) &= 3c_1\theta + c_2 + 2\rho\sin(\theta+\psi) \\ p_2(\theta) &= 2c_1 + \rho\cos(\theta+\psi) \\ p_3(\theta) &= \alpha\sin(\theta+\Psi) + \beta\cos(\theta+\Psi) \end{aligned} \tag{47}$$

Writing Eq. (45) in terms of its components, the necessary and sufficient condition for singular solutions is that

$$p_1(\theta)^2 + p_2(\theta)^2 + p_3(\theta)^2 = b_1^2/4 \tag{48}$$

is satisfied identically on Θ. Substituting Eq. (47) into Eq. (48) and solving for the constants, two types of singular solutions are found. They are analogous to the two types of singular solutions presented previouly and additional details can be inferred from Ref. 37.

1. Chattering in the Orbital Plane

If the motion is confined to the plane of the circular orbit, then $\alpha = \beta = 0$ and Eqs. (47) and (48) simplify. There are two solutions found; $c_1 = 0, c_2 = \pm b_1/2, \rho = 0$, and ψ arbitrary. The primer vector (44) becomes

$$p(\theta) = \left(\pm\frac{b_1}{2}, 0, 0\right)^T$$

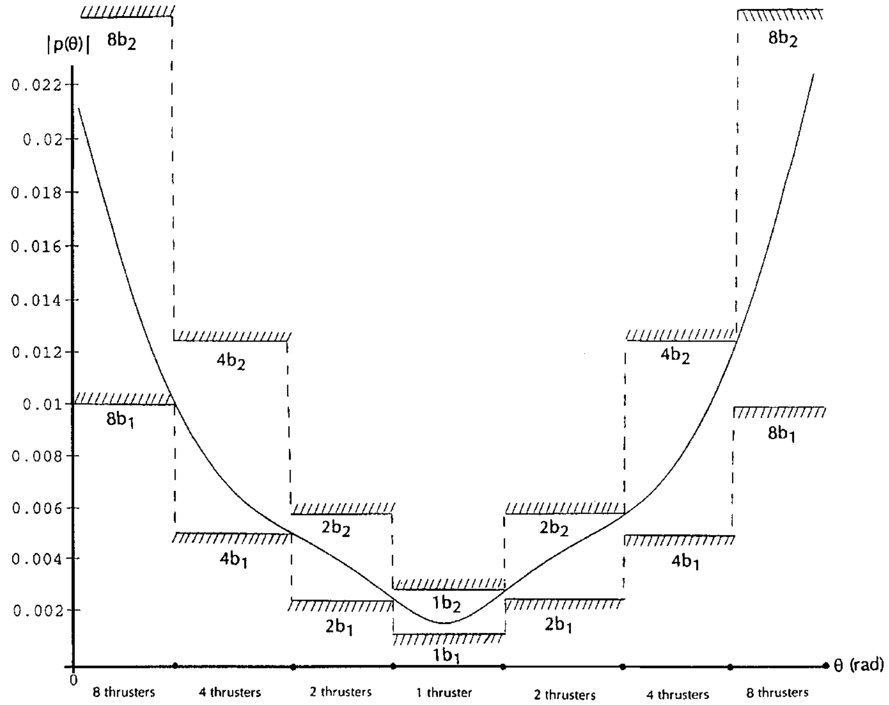

Fig. 3a Primer vector magnitude and burn sequence of eight thrusters for unsaturated flight: b_1 = 0.00125 and b_2 = 0.003.

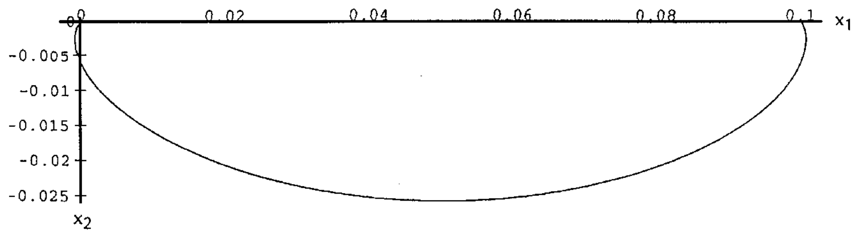

Fig. 3b Flight path for spacecraft with eight thrusters: $x_1(0)$ = 0.1, $x_2(0)$ = 0, b_1 = 0.00125, and b_2 = 0.003.

In one solution the control $\boldsymbol{u}(\theta)$ can chatter between $(0, 0, 0)^T$ and $(-b_1/2, 0, 0)^T$). In the other $\boldsymbol{u}(\theta)$ can chatter betweeen $(0, 0, 0)^T$ and $(b_1/2, 0, 0)^T$. Although there are only two primer vectors of this type, there is a class of boundary conditions z_f associated with them, and a variety of nonunique chattering controls associated with a typical boundary condition.

2. Chattering Out of the Orbital Plane

There are also two solutions for the constants if the motion is out of the orbital plane, that is, if α and β are not both zero. In this case $c_1 = c_2 = \alpha = 0$, $\rho = b_1/4$, $\beta = \pm\sqrt{3}b_1/4$, and ψ is arbitrary. The primer vector is given by

$$p(\theta) = (b_1/2)\left[\sin(\theta+\psi), \tfrac{1}{2}\cos(\theta+\psi), \pm(\sqrt{3}/2)\cos(\theta+\psi)\right]^T$$

In one solution $\boldsymbol{u}(\theta)$ can chatter between the origin and the circular arc $-b_1/2[\sin(\theta+\psi), \frac{1}{2}\cos(\theta+\psi), \sqrt{3}/2\cos(\theta+\psi)]^T$. In the other $\boldsymbol{u}(\theta)$ can chatter between the origin and the circular arc $-b_1/2[\sin(\theta+\psi), \frac{1}{2}\cos(\theta+\psi), -(\sqrt{3}/2)\cos(\theta+\psi)]^T$. There are only two primer vectors of this type, but associated with each is a class of boundary values z_f. A typical boundary value z_f associated with chatter has many chattering controls associated with it.[37] The two classes of chattering controls associated with this type rotate with constant angular speed completing a rotation during one orbital period, and are confined to a plane inclined ±60 deg with respect to the orbital plane.

C. Computer Simulations

For the normalized equation (43) and various distinct settings of the thrust magnitude bounds b_1 and b_2 and various initial conditions, the nonlinear equation (26) was solved numerically by Newton's method to compute rendezvous trajectories of the spacecraft. The initial guess was obtained by removing the upper and lower constraints on the thrust magnitude and solving for $\boldsymbol{c}$ in closed form as indicated in previous work.[20] The constraints were then incorporated and Eq. (26) was solved by Newton's method; the constraints were tightened and the procedure repeated several times to produce the results presented in the following figures. The convergence process was found to be more sensitive to the lower bound than to the upper one. The final results are almost certainly correct because the values of $\boldsymbol{c}$ obtained satisfy Eq. (26) to within a very small tolerance, and the computed control functions produce trajectories that also match the terminal boundary conditions within a very small tolerance.

The objective of the rendezvous was to match the terminal position and velocity of the spacecraft with those of the satellite. From these studies two thrust-saturated rendezvous simulations and one multiple-thruster unsaturated rendezvous simulation are presented. In each case the rendezvous duration is one orbit (e.g., $\theta_0 = 0$, $\theta_f = 2\pi$), and the boundary conditions confine the trajectory to the plane of the orbit. Since the plotter connected the points by straight lines, some of the figures show corners that are fictitious. The worst of these is seen in the thrust plots in Figs. 1b and 2b.

1. Single Thruster Solutions with Upper and Lower Saturation

The first simulation depicts a problem in which the optimal solution requires both upper and lower thrust saturation without engine shutoff. The results are presented in Fig. 1. In this study the spacecraft is originally 0.2 normalized units above the satellite with the same velocity as the satellite. The thrust saturation bounds are $b_1 = 0.055$ and $b_2 = 0.250$.

For these conditions, the solution of Eq. (26) defines the primer vector whose locus is presented in Fig. 1a. It moves clockwise beginning at the left, having outer thrust saturation, and pierces the circle defined by the outer bound b_2 causing an interval of unsaturated thrusting, then pierces the inner circle of radius b_1 resulting in an interval of inner thrust saturation, moves back into the annular region of unsaturated thrusting and, finally, leaves the outer circle moving to the right in an interval of outer thrust saturation. The resulting thrust vector locus is seen in Fig. 1b, where the thrust vector moves clockwise beginning at the right and ending at the left depicting the various phases of saturation and unsaturation. The corresponding flight path is presented in Fig. 1c.

A different situation is found in Fig. 2, where a short interval of engine shutoff occurs. Here the spacecraft begins 0.2 normalized units behind the satellite with the same velocity, and the thrust bounds are $b_1 = 0.006$ and $b_2 = 0.010$.

It is observed from the primer vector locus of Fig. 2a that the primer vector, moving clockwise, goes inside both circles passing near enough to the origin to cause engine shutoff. Both outer and inner saturation of the thrust can be seen in Fig. 2b, as the thrust vector begins at the right, moves clockwise, and ends at the left. The straight line segment from the left to the origin represents instantaneous engine shutoff. After remaining at the origin for an interval of time, the engine ignites instantaneously as indicated by the straight line segment from the origin directed to the right. The resulting flight path is shown in Fig. 2c.

2. *Multiple-Thruster Unsaturated Solution*

Finally, an example is presented in Fig. 3, in which the use of multiple thrusters with the capability of shutting on and off independently leads to a totally unsaturated solution. In this example, the spacecraft is initially 0.1 normalized units behind the satellite having the same velocity as the satellite. The thrust bounds are $b_1 = 0.00125$ and $b_2 = 0.003$.

Considering first the upper bound b_2 only, an application of the efficiency condition shows that at least eight thrusters are necessary for an unsaturated solution [i.e., $\nu = 8$ is the smallest integer that satisfies the right-hand side of Eq. (35)]. The question of lower saturation cannot be resolved by corollaries 1 or 2 because neither Eq. (41) nor Eq. (42) is satisfied. The theorem, however, shows that a multiple-thruster unsaturated solution does exist, after examining the graph of the magnitude of the primer vector (40). This is depicted in Fig. 3a, which shows a plot of this primer vector magnitude and a thruster burn sequence function j that satisfies the condition (38). The burn sequence consists of seven phases: first eight thrusters burn, then four, then two, then only one, then two, then four, and finally eight again. This leads to a totally unsaturated solution whose flight path is presented in Fig. 3b. This type of burn sequence cannot occur if $b_2/b_1 < 2$. For details see Ref. 35.

V. Conclusion

If upper and lower bounds are placed on the thrust magnitude, the solution of the optimal power-limited rendezvous problem consists of four admissible states: thrusting at the upper saturation level, unsaturated thrusting, thrusting at the lower saturation level, and engine shutoff. There is, additionally, a fifth singular state that can involve chattering for some boundary conditions. For the example of linearized rendezvous near circular orbit, exactly four classes of chattering solutions can appear but they are of two types; one type is confined to the plane of the orbit, the other is not. The greatest fuel efficiency is found if the solution is totally unsaturated. If the spacecraft and thrusters are designed so that multiple thrusters can be mounted and can burn independently, totally unsaturated solutions are possible for many problems. A necessary and sufficient condition for multiple-thruster unsaturated solutions is presented with two corollaries for the case where the equations of motion are linear. For the case of rendezvous near circular orbit, single-thruster computational examples with and without coasting intervals, and a multiple-thruster unsaturated example having a seven-phase burn sequence are presented.

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